Exercise 71

Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Solution

The rational function $1/x$ is continuous at all $x \neq 0$ by Theorem 7, the sine of $1/x$ is continuous at all $x \neq 0$ by Theorem 9, x^4 is continuous at all $x \neq 0$ by Theorem 7, and the product $x^4 \sin(1/x)$ is continuous at all $x \neq 0$ by Theorem 4. All that's left to show is that there's continuity at the point where the function changes:

$$\lim_{x \to 0} f(x) = f(0),$$

that is,

$$\lim_{x \to 0} x^4 \sin(1/x) = 0.$$

Evaluate the left side.

$$\begin{aligned} \lim_{x \to 0} x^4 \sin\left(\frac{1}{x}\right) &= \lim_{x \to 0} \frac{x^3}{\frac{1}{x}} \sin\left(\frac{1}{x}\right) \\ &= \lim_{x \to 0} x^3 \left[\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right] \\ &= \left(\lim_{x \to 0} x^3\right) \left[\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right] \end{aligned}$$

For the limit in square brackets, make the substitution $u = 1/x$. Then as $x \to 0$, $u \to \pm\infty$.

$$\begin{aligned} \lim_{x \to 0} x^4 \sin\left(\frac{1}{x}\right) &= \left(\lim_{x \to 0} x^3\right) \left(\lim_{u \to \pm\infty} \frac{\sin u}{u}\right) \\ &= \left(\lim_{x \to 0} x\right)^3 \left(\lim_{u \to \pm\infty} \frac{\sin u}{u}\right) \\ &= (0)^3(0) \\ &= 0 \end{aligned}$$

Therefore, $f(x)$ is continuous on $(-\infty, \infty)$.